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Asymptotic solutions for nonlinear thermal convection in porous media

B. KOCHHAR and B. C. CHANDRASEKHARA
Department of Physics, Central College, Bangalore University, Bangalore 560001, India

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Abstract—Nonlinear thermal convection in a saturated porous medium is studied under different approximations. Asymptotic solutions for free boundaries are obtained for two limiting cases: Ra large, $\sigma_1 = O(1)$ and $O(Ra)$ using the method of matched asymptotic expansions. The analytical approximation method is also used to study the convection problem over a wide range of Ra and σ_1. The results of the analysis show that the heat transfer rate varies as $Ra^{1/3}$ for a given σ_1. The results are in agreement with the experimental results of Schneider.

INTRODUCTION

In recent years there has been considerable interest given to the study of convection problems in a porous medium because of its importance in many industrial and geophysical problems (namely geo-thermal energy extraction, reactor technology and cryogenic industry). An important physical quantity in such studies is the determination of the Nusselt number, which is a measure of the total convective and conductive heat fluxes. The aim of this paper is to study the variation of Nusselt number with the parameters of the problem, namely, porous parameter and Rayleigh number.

The solution of the governing equations are obtained using the mean field approximation. In this approach one defines mean quantities as horizontal averages and decomposes all the quantities into mean and fluctuating parts. Further, the mean field equations are obtained by omitting the terms which are nonlinear in fluctuating quantities. The above approach has successfully been followed by Herring [1, 2], Howard [3], Roberts [4] and Van der Borght [5, 6] to study convection problems. The analysis is extended to two situations: (1) for large Ra, in which asymptotic solutions are obtained for two limiting cases (a) $Ra \to \infty$, $\sigma_1 = O(1)$, and (b) Ra large, $\sigma_1 = O(Ra)$; (2) for a wide range of Ra and σ_1. As in most discussions of mean field equations, the present work is confined to only one horizontal mode.

The results of analysis show that the Nusselt number varies as $Ra^{1/3}$ for a given value of σ_1. The results are in agreement with those of Schneider [7] which is one of the many experimental studies on convection in porous media. However, we note that considerable disagreement exists among the many experiments available [7–9]. The experimental results of Masuoka [8] and Elder [9] suggest a linear increase in Nusselt number for large Ra. An estimate of the critical Rayleigh number is also obtained from the relation for Nusselt number which is in agreement with the results obtained by Malkus and Veronis [10] for $\sigma_1 = 0$ and those obtained by Lapwood [11] and Masuoka [12] for porous media.

MATHEMATICAL FORMULATION

For mathematical analysis we consider a horizontal slab of saturated porous layer between planes $z = 0$ and d, where z is the vertical coordinate. The boundary temperatures are assumed to be fixed with the lower boundary at a higher temperature and the temperature difference, ΔT, across the layer is constant. The thickness, d, of the layer is assumed to be sufficiently small that the Boussinesq approximation holds good. Further, we use the Brinkman momentum equation which gives the complete description of the fluid in a porous medium. As usual, we decompose the temperature into mean and fluctuating parts, $T = \langle T\rangle + \theta$, where angular brackets indicate the horizontal average. The non-dimensional equations of motion after averaging over the horizontal plane become [4, 13]

$$\frac{1}{\sigma}\left[\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}-\langle\mathbf{u}\cdot\nabla\mathbf{u}\rangle\right] = -\nabla\bar{\omega}+Ra\,\hat{k}\theta+\nabla^2\mathbf{u}-\sigma_1\mathbf{u}, \quad (1)$$

$$\frac{\partial\langle T\rangle}{\partial t} = -\frac{\partial\langle w\theta\rangle}{\partial z}+\frac{\partial^2\langle T\rangle}{\partial z^2}, \quad (2)$$

$$\frac{\partial\theta}{\partial t}+\mathbf{u}\cdot\nabla\theta-\langle\mathbf{u}\cdot\nabla\theta\rangle = \beta w+\nabla^2\theta, \quad (3)$$

$$\nabla\cdot\mathbf{u} = 0, \quad (4)$$

where

$$\beta = -\frac{\partial\langle T\rangle}{\partial z}, \quad (5)$$

and $Ra = g\alpha'\Delta T d^3/\nu k$, $\sigma = \nu/k$ and $\sigma_1 = d^2/K$ are Rayleigh number, Prandtl number and porous parameters, respectively. Eliminating $\bar{\omega}$ and the horizontal components of velocity and neglecting the terms which are nonlinear in fluctuating quantities, the mean field equations for steady convection become

$$\nabla^4 w-\sigma_1\nabla^2 w+Ra\,\nabla_1^2\theta = 0, \quad (6)$$

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NOMENCLATURE

A, A_1, A_3	coefficients defined by equations (26), (24a) and (24b), respectively
a	horizontal wave number
B_1, B_3	coefficients defined by equations (39a) and (39b)
$C(\alpha)$	heat transport shape factor, equation (61)
D	$\partial/\partial z$
d	thickness of the layer
F	defined by equation (15)
f	periodic function in the horizontal plane defined by equation (10)
G	defined by equation (55)
g	acceleration due to gravity
J_n	coefficients defined by equation (46)
K	permeability
k	thermal diffusivity
$\hat{k}$	unit vector in z-direction
Nu	Nusselt number
P	defined by equation (19), $Ra\, a^2\, Nu$
Ra	Rayleigh number, $g\alpha'\Delta T d^3/\nu k$
s	defined by equation (62)
T	temperature
u, v, w	x, y, z components of velocity $\mathbf{u}$
W	defined by equation (9a)
w_n	coefficients defined by equation (42a)
x	α^2
x_p	coefficients defined by equation (57)
$X(\alpha)$	defined by equation (62)

Greek symbols

α	a/π
α'	coefficient of thermal expansion
β	$-\partial\langle T\rangle/\partial z$
β_n	coefficients defined by equation (42c)
γ	$\lambda a^2 Nu$
ε	$\varepsilon = [2G/J_1(Nu-1)\tau^2]^{1/2}(\alpha^2/8)$
Θ	temperature defined by equation (9b)
θ_n	coefficients defined by equation (42b)
θ	fluctuations in mean temperature
λ	Ra/σ_1
ν	kinematic viscosity
ξ	scaled z coordinate, equation (25)
ρ	defined by equation (51)
σ	Prandtl number, ν/k
σ_1	porous parameter, d^2/K
τ_n	defined by equation (52)
τ, ρ	defined by equation (56)
ϕ	scaled temperature field, equation (34)
ψ	scaled velocity field, equations (19) and (27)
Ω	scaled velocity field defined by equation (34)
$\bar{\omega}$	deviation of pressure from hydrostatic value divided by density.

Other symbols

∇_1^2	operator, $\partial^2/\partial x^2+\partial^2/\partial y^2$
$\langle\ \rangle$	horizontal average.

$$\frac{\partial\langle w\theta\rangle}{\partial z} = \frac{\partial^2\langle T\rangle}{\partial z^2}, \tag{7}$$

$$-\beta w = \nabla^2\theta, \tag{8}$$

where

$$\nabla_1^2 = \partial^2/\partial x^2+\partial^2/\partial y^2.$$

The basic equations are separable and admit solutions of the forms

$$w = f(x,y) \quad W(z,t), \tag{9a}$$

$$\theta = f(x,y) \quad \Theta(z,t), \tag{9b}$$

where function f satisfies the plan-form equation with a as a separation constant

$$\nabla_1^2 f = -a^2 f. \tag{10}$$

The basic equations, equations (6) and (8), after incorporating equations (9a) and (9b) reduce to the form

$$(D^2-a^2)^2W-\sigma_1(D^2-a^2)W-Ra\,a^2\Theta = 0, \tag{11}$$

$$(D^2-a^2)\Theta = -\beta W. \tag{12}$$

Equation (7) leads to the definition of non-dimensional heat transfer or Nusselt number

$$Nu = \beta+W\Theta. \tag{13a}$$

On integrating equation (13a) over z we get

$$Nu = 1+\int_0^1 W\Theta\,dz. \tag{13b}$$

For free boundaries the velocity and isothermal boundary conditions are

$$\left.\begin{aligned} W = D^2W = 0 \quad &\text{at} \quad z = 0,1, \\ \Theta = 0 \quad &\text{at} \quad z = 0,1. \end{aligned}\right\} \tag{14}$$

ASYMPTOTIC SOLUTIONS FOR LARGE RAYLEIGH NUMBER

The solutions of the mean field equations for large Ra are obtained for two limiting cases, namely, (1) when $\sigma_1 = O(1)$ and (2) $\sigma_1 = O(Ra)$, using the method of matched asymptotic expansions.

Case (a): $\sigma_1 = O(1)$ and Ra is large

Following Van der Borght [6] we assume that the horizontal wave number $a = O(1)$. Introducing the variable F defined by

$$F = Nu^{-1}(\Theta) \tag{15}$$

in equations (11) and (12) we get

$$(D^2-a^2)^2W-\sigma_1(D^2-a^2)W-Ra\,a^2F\,Nu = 0, \tag{16}$$

$$(D^2-a^2)F = -(1-FW)W. \tag{17}$$

Combining equations (11), (13a) and (12) we get

$$(D^2-a^2)^3W-\sigma_1(D^2-a^2)^2W + Ra\,a^2Nu\,W-W^2(D^2-a^2)^2W + \sigma_1 W^2(D^2-a^2)W = 0. \tag{18}$$

Since $Nu \geqslant 1$ ($Nu \leqslant 1$ is the conduction region and $Nu > 1$ is the convection region) and $a = O(1)$ we find that $Ra\,a^2\,Nu \to \infty$ as $Ra \to \infty$. Away from boundaries (i.e. in the interior region) we expect $D = \partial/\partial z = O(1)$ and therefore, as $Ra \to \infty$, terms like $Ra\ a^2\ Nu$ in equation (18) become large requiring W to be large for a nontrivial solution of equation (18) to exist. This condition is met by choosing $W = O(P^{1/2})$ where $P = Ra\ a^2\ Nu$. Defining the variable

$$\psi = (Ra\,a^2\,Nu)^{-1/2}W = P^{-1/2}W, \tag{19}$$

we find that the equation which governs the motion in the main body of the fluid to leading order is

$$\psi(D^2-a^2)^2\psi-\sigma_1\psi(D^2-a^2)\psi = 1, \tag{20}$$

and the transformed boundary conditions are

$$\psi = D^2\psi = 0 \quad \text{at} \quad z = 0, 1. \tag{21}$$

For finding an approximate solution of equation (20) we use a truncated Fourier series for ψ in the form [3]

$$\psi = A_1 \sin \pi z + A_3 \sin 3\pi z +, \ldots \tag{22}$$

This solution is symmetric about $z = 1/2$ and satisfies the free boundary conditions. Substitution of this series solution in equation (20) and retaining the first two terms of the expansion gives

$$\begin{aligned} &A_1[(\pi^2+a^2)^2+\sigma_1(\pi^2+a^2)]\sin \pi z \\ &\quad + A_3[(9\pi^2+a^2)^2+\sigma_1(9\pi^2+a^2)]\sin 3\pi z \\ &= \frac{1}{A_1 \sin \pi z}\left[1-\frac{A_3}{A_1}\frac{\sin 3\pi z}{\sin \pi z}\right] \\ &= \frac{1}{A_1 \sin \pi z}\left[1-\frac{A_3}{A_1}(1+\cos 2\pi z)\right]. \end{aligned} \tag{23}$$

To determine A_1 and A_3 we multiply equation (23) by $\sin \pi z$ and $\sin 3\pi z$, separately and integrate over z. This leads to two equations which are solved assuming $A_3/A_1 \ll 1$. The assumption $A_3/A_1 \ll 1$ is valid as long as the sine series is a good approximation. The expressions for A_1 and A_3 are

$$A_1 \simeq \left[\frac{2}{\{(\pi^2+a^2)^2+\sigma_1(\pi^2+a^2)\}}\right]^{1/2} \times \left[1-\frac{\{(\pi^2+a^2)^2+\sigma_1(\pi^2+a^2)\}}{2\{(9\pi^2+a^2)^2+\sigma_1(9\pi^2+a^2)\}}\right], \tag{24a}$$

and

$$A_3 \simeq \left[\frac{(\pi^2+a^2)^2+\sigma_1(\pi^2+a^2)}{(9\pi^2+a^2)^2+\sigma_1(9\pi^2+a^2)}\right]A_1. \tag{24b}$$

The determination of A_1 and A_3 completes the outer solution or approximation to ψ in the interior of the fluid. F is determined by observing that the relation $F = 1/W$ satisfies equation (17) to the leading order for large W. The same conclusion is drawn by comparing equations (15) and (20).

For finding the inner solution or the solution in the boundary layer, we use the variable

$$\xi = P^{1/4}z. \tag{25}$$

To obtain the appropriate matching condition we note that near $z = 0$ equations (22), (24a), and (24b) imply $\psi \sim Az$ where

$$A \simeq (A_1+3A_3). \tag{26}$$

Introducing the variables

$$W = P^{1/4}\psi \quad \text{and} \quad F = P^{-1/4}f, \tag{27}$$

equations (16) and (17) become

$$P^{1/2}\frac{\partial^4\psi}{\partial\xi^4} = f+\sigma_1\frac{\partial^2\psi}{\partial\xi^2}, \tag{28}$$

and

$$\frac{\partial^2 f}{\partial\xi^2} = -(1-f\psi)\psi. \tag{29}$$

The transformed boundary conditions are

$$\psi = \frac{\partial^2\psi}{\partial\xi^2} = f = 0 \quad \text{at} \quad \xi = 0, 1. \tag{30}$$

Following the analysis of Howard [3] we obtain the expression for the Nusselt number in the form

$$Nu = \left[\frac{2\Gamma(3/4)}{2}\right]^{-4/3}(Aa^2)^{2/3}Ra^{1/3}, \tag{31}$$

$$Nu \simeq \left[\frac{1}{2.124}\right]^{4/3}\left[\frac{2\alpha^2}{\{(1+\alpha^2)^2+(1+\alpha^2)\sigma_1/\pi^2\}}\right]^{1/3} \times \left[1+\frac{5}{3}\frac{\{(1+\alpha^2)^2+(1+\alpha^2)\sigma_1/\pi^2\}}{\{(9+\alpha^2)^2+(9+\alpha^2)\sigma_1/\pi^2\}}\right]Ra^{1/3}, \tag{32}$$

where $\alpha = a/\pi$.

Case (b): Ra large, $\sigma_1 = O(Ra)$

The analysis in Case (a) was carried out for $\sigma_1 = O(1)$. We will now consider the case $\sigma_1 = O(Ra)$ which is of practical importance since σ_1 is usually of the order of 10^3 or more. For this case, we define

$$\lambda = Ra/\sigma_1 \quad \text{and} \quad \gamma = \lambda a^2\,Nu. \tag{33}$$

We note that when the stabilizing effect due to σ_1 increases, $Nu \to 1$. Hence, in that part of the regime $\lambda = O(1)$ although Ra and σ_1 are both large.

Introducing the scalings

$$W = \left(\frac{Ra\,a^2\,Nu}{\sigma_1}\right)^{1/2}\Omega \quad \text{and} \quad F = \left[\frac{\sigma_1}{Ra\,a^2\,Nu}\right]^{1/2}\phi, \tag{34}$$

equations (11) and (12) become

$$(D^2-a^2)\Omega+\phi = 0, \tag{35}$$

$$(D^2-a^2)\phi = -\gamma(1-\phi\Omega)\Omega. \tag{36}$$

After eliminating ϕ from equations (35) and (36) we get

$$D^4\Omega-(2a^2+\Omega^2\gamma)D^2\Omega+(a^4+a^2\Omega^2\gamma-\gamma)\Omega = 0. \tag{37}$$

The solution of the above equation in analogy with the previous one is

$$\Omega = B_1\sin\pi z+B_3\sin 3\pi z+,\ldots \tag{38}$$

Substituting the above expansion in equation (37) and multiplying it by $\sin\pi z$, $\sin 3\pi z$, separately and integrating with respect to z from 0 to 1 we get

$$4(\pi^4+2\pi^2a^2+a^4-2\gamma)$$
$$= -B_1^2\pi^2\gamma[(3+3a^2/\pi^2)-(11+3a^2/\pi^2)B_3/B_1], \tag{39a}$$

$$4B_3(81\pi^4+18\pi^2a^2+a^4-2\gamma)$$
$$= -B_1^3\pi^2\gamma[-(1+a^2/\pi^2)+(22+6a^2/\pi^2)B_3/B_1]. \tag{39b}$$

Likewise if we expand ϕ as

$$\phi = \alpha_1\sin\pi z+\alpha_3\sin 3\pi z+,\ldots, \tag{40}$$

and substitute the values of Ω and ϕ in equation (36), then we obtain the solution for α_1 and α_3. This completes the outer or the interior solution. The boundary layer solution need not be treated separately because the outer solutions satisfy all the boundary conditions and give an adequate description of the entire solution. The Nusselt number is computed from the relation

$$Nu^{-1} = \int_0^1 (1-\phi\Omega)\,dz,$$
$$= \left[1-\frac{1}{2}\alpha_1B_1(1+\alpha_3B_3/\alpha_1B_1)\right]. \tag{41}$$

SOLUTION FOR WIDE RANGE OF *Ra* AND σ_1

The analysis carried out in the previous section gives an asymptotic solution for two limiting cases. However, to study the behaviour of Nusselt number over a wider range of Ra and σ_1 we use the analytical method given by Herring [2]. The method is accurate for all values of Ra provided α, the horizontal wave number which supports the convection, is not too large. This method is also applicable for rigid boundaries although it is more involved [2].

Using the following expansions for w, θ and β

$$w = \pi\sqrt{2}\sum_{n=1}^{\infty} f(x,y)w_n\sin n\pi z, \tag{42a}$$

$$\theta = \frac{\sqrt{2}}{\pi}\sum_{n=1}^{\infty} f(x,y)\theta_n\sin n\pi z, \tag{42b}$$

$$\beta = 1+\sum_{n=1}^{\alpha}\beta_n\cos 2n\pi z, \tag{42c}$$

in equations (6)–(8) we get a set of nonlinear algebraic equations

$$w_n = J_n\theta_n, \tag{43}$$

$$\beta_n = -\sum_p w_{2p-1}[\theta_{2(n+p)-1}$$
$$+Y(2p-2n-1)\theta_{|(2n-2p+1)|}], \tag{44}$$

$$\theta_{2n-1} = \frac{T_n}{2}\sum_p[\beta_{|n-p|}-\beta_{n+p-1}]w_{2p-1}, \tag{45}$$

where

$$J_n = \frac{Ra\,\alpha^2}{\{(n^2+\alpha^2)^2+(n^2+\alpha^2)\sigma_1/\pi^2\}\pi^4}, \tag{46}$$

$$T_n = \frac{1}{\{(2n-1)^2+\alpha^2\}}, \tag{47}$$

$$\beta_0 = 2,$$

and

$$Y(X) = 1, \quad X>0,$$
$$= 0, \quad X = 0,$$
$$= -1, \quad X<0.$$

The Nusselt number is obtained from equation (44) by summing over n, i.e.

$$N = \sum\beta_n+1. \tag{48}$$

In writing equations (44) and (45) advantage has been taken of the symmetry of the problem hence only those amplitudes which are non-zero in the stable steady state are recorded. Equations (44) and (45) can be written as

$$\rho_n\beta_n = w_1(\theta_{2n-1}-\theta_{2n+1}), \tag{49}$$

$$\theta_{2n-1} = \frac{1}{2}J_1\theta_1\tau_nT_n(\beta_{n-1}-\beta_n), \tag{50}$$

where

$$\rho_n = \sum_{p=1}^{\infty}\frac{(\theta_{2n-1}-\theta_{2n+1})J_1\theta_1}{(\theta_{|2n-2p+1|}Y(2n-2p+1)-\theta_{2n+2p-1})J_{2p-1}\theta_{2p-1}} \tag{51}$$

and

$$\tau_n = \sum_{p=1}^{\infty}\frac{(\beta_{|n-p|}-\beta_{n+p-1})}{(\beta_{n-1}-\beta_n)}\,\frac{J_{2p-1}\theta_{2p-1}}{J_1\theta_1}. \tag{52}$$

By substituting equation (49) in equation (50) we get

$$\tau_{n+1}T_{n+1}\beta_{n+1}+\tau_nT_n\beta_{n-1}$$
$$-\left(\frac{2G\rho_n}{J_1(Nu-1)}+\tau_nT_n+\tau_{n+1}T_{n+1}\right)\beta_n = 0, \tag{53}$$

and equation (49) summed from $n = 0$ to ∞ leads to

$$\theta_{2n-1} = \theta_1 G \sum_{p=n}^{\infty} \frac{\beta_p \rho_p}{(Nu-1)}, \tag{54}$$

where

$$G = \sum_{n=1}^{\infty} \frac{w_{2n-1}\theta_{2n-1}}{w_1\theta_1}. \tag{55}$$

An iterative procedure can be used to find β_n's and hence Nu from equations (53) and (48). Once β_n's are known θ_n's are found from equation (54). However, when Ra is large we can use a continuous approximation for β_n and θ_n spectra. In this approximation equations (51) and (52) reduce to

$$\tau_n \to \sum_p \frac{J_{2p-1}}{J_1} x_{2p-1} \simeq 1/\rho_n, \tag{56}$$

where

$$x_{2p-1} = (2p-1)\frac{\theta_{2p-1}}{\theta_1}. \tag{57}$$

In deriving equation (56) terms like $(\beta_{|n-p|} - \beta_{n+p-1})$ are approximated by $(2p-1)(\beta_n - \beta_{n-1})$ which makes τ_n and ρ_n independent of n, hereafter referred as τ and ρ. This approximation is valid when Nu is fairly large ($Nu \geqslant 2$). Equation (53) reduces to

$$T_{n+1}\beta_{n+1} + T_n\beta_{n-1} - \left[\frac{2G}{J_1\tau^2(Nu-1)} + T_n + T_{n+1}\right]\beta_n = 0. \tag{58}$$

When we consider continuous spectra for β_n's, the summation can be replaced by an integral and the final expression for Nu is given by solving equation (58) [2]

$$Nu = 1 + \frac{1}{2}\left[\frac{\Gamma(1/4+\varepsilon/2)}{\Gamma(3/4+\varepsilon/2)}\right]^{4/3}\left[\frac{Ra}{\pi^4}\right]^{1/3}\left[\frac{\beta_1}{2}\right]^{4/3} C(\alpha),$$

where

$$\varepsilon = \left[\frac{2G}{J_1(Nu-1)\tau^2}\right]^{1/2}\frac{\alpha^2}{8}.$$

When $\varepsilon \to 0$

$$Nu - 1 \simeq 0.4617\left(1 - \frac{Ra(\alpha)}{Ra}\right)^{4/3} C(\alpha) Ra^{1/3}, \tag{59}$$

where

$$Ra(\alpha) = \frac{\pi^4(1+\alpha^2)\{(1+\alpha^2)^2 + (1+\alpha^2)\sigma_1/\pi^2\}}{\alpha^2}, \tag{60}$$

and

$$C(\alpha) = \left[\frac{\pi^4 J\tau^2}{Ra\,G}\right]^{1/3}. \tag{61}$$

$C(\alpha)$ is called the heat transport shape factor. When α is small $x_p(\alpha)$ will decrease rather slowly and to obtain τ and G they can be approximated by a constant value $X(\alpha)$ given by

$$X(\alpha) = -\frac{s}{2} + \left[\frac{s^2}{4} + s\right]^{1/2}, \tag{62}$$

where

$$\frac{1}{s} = \sum_{p=1}^{\infty} (2p-2)\frac{J_{2p-1}}{J_1}.$$

In the limit $\sigma_1 \to 0$ (in the absence of porous media) equation (59) reduces to that obtained by Herring [2].

ESTIMATION OF CRITICAL RAYLEIGH NUMBER

An approximate estimate of the critical Rayleigh number can be obtained from equation (59) for different values of σ_1 by substituting $Nu = 1$, we get $Ra = Ra(\alpha)$. The minimum value of Ra for a given σ_1 occurs at $\alpha = \alpha_c$, the critical wave number determined by putting $dRa/d\alpha = 0$. In equation (60) τ also depends on α, as a first approximation we take it as a constant equal to 1. α_c is determined from the following cubic equation

$$2\pi^2 x'^3 + (3\pi^2 + \sigma_1)x'^2 - (\pi^2 + \sigma_1) = 0, \tag{63}$$

where $x' = \alpha^2$. For $\sigma_1 = 0$ the critical Rayleigh number $Ra_c = 657.8$ is attained at $a_c = 2.221$ which is exactly the value given by Malkus and Veronis [10]. When σ_1 is large the first term of equation (60) is negligible as compared to the second term, hence

$$Ra = \frac{\pi^2(1+\alpha^2)^2\sigma_1}{\alpha^2}. \tag{64}$$

For $\alpha = 1$, $Ra = 4\pi^2\sigma_1$, which is the same as given by Lapwood [11]. But in the actual problem $\tau \neq 1$ but it is different for different α. One way of incorporating τ is to find the value of α_c from equation (63) and for that value of α_c find τ and hence Ra. The critical Rayleigh numbers

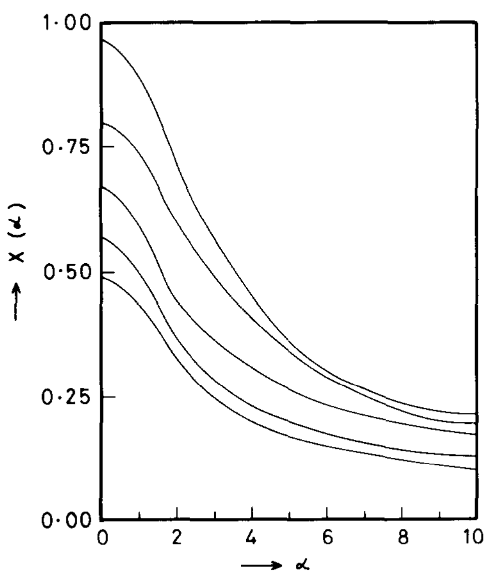

FIG. 1. $X(\alpha)$ vs α for $\sigma_1 = 0, 10^2, 10^3, 10^4, 10^5$, and $Ra = 10^7$.

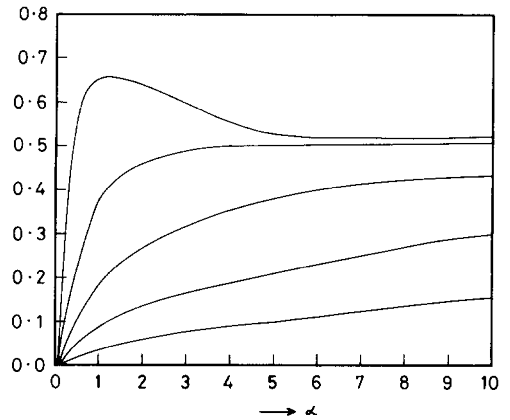

FIG. 2. $C(\alpha)$ vs α for $\sigma_1 = 0, 10^2, 10^3, 10^4, 10^5$. and $Ra = 10^7$.

obtained are given below and are compared with results obtained by earlier workers. For $\sigma_1 = 100$, $Ra_c = 4699$ at $a_c = 2.923$ whereas, the value obtained by Masuoka [12] is 4660. Rudraiah and Rohini [14] using the method of Pellow and Southwell found $Ra_c = 4719.15$ at $a_c = 2.768$ for the above value of σ_1. The critical Rayleigh number for $\sigma_1 = 10^3$ and 10^4 are found to be 49 254 and 394 851, respectively, attained at $a_c = 3.112$ and 3.139.

RESULTS AND DISCUSSIONS

Figure 1 gives the values $X(\alpha)$ vs α calculated from equation (61). The values of $X(\alpha)$ decrease as σ_1 increases. Since $X(\alpha) = x_n \equiv \theta_n/\theta_1$, hence an increase in σ_1 results in stabilizing the temperature field. Consequently the Nusselt number will decrease with increasing σ_1.

Figure 2 shows the variation of heat transport shape factor $C(\alpha)$ with α. For a given value of σ_1 heat convection attains a constant value at large α and is independent of α. From equation (59) we obtain that for

$$\alpha \geqslant \alpha_0 = \left[\frac{-\sigma_1/\pi^2 + (\sigma_1/\pi^2 + 4Ra/\pi^4)^{1/2}}{2}\right]^{1/2}$$

the convective heat transport is zero.

Figure 3 indicates the variation of Nusselt number with σ_1 for various values of Ra for $a = \pi$. The values for $\sigma_1 = O(1)$ are obtained from equation (32). To calculate Nu for $\sigma_1 = O(Ra)$ we use the iterative procedure in which we take a trial value of Nu, compute B_3/B_1 and B_1 from equations (39a) and (39b) and hence α_1, α_3/α_1, recompute Nu from equation (41) till a constant value of Nu is obtained. Figure 3 also includes the values of Nu obtained using equation (59). The results obtained from the limiting cases $\sigma_1 = O(1)$ and $O(Ra)$ agree with the results obtained from equation (59). Hence, the analytical approximation method discussed for a wide range of Ra and σ_1 gives accurate results for Nu for the whole range of σ_1. We have also used the same procedure to study the convection problem in the

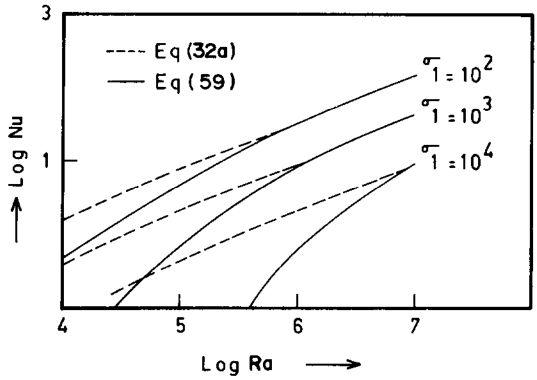

FIG. 4. Log Nu vs log Ra for three values of σ_1, with $a = \pi$.

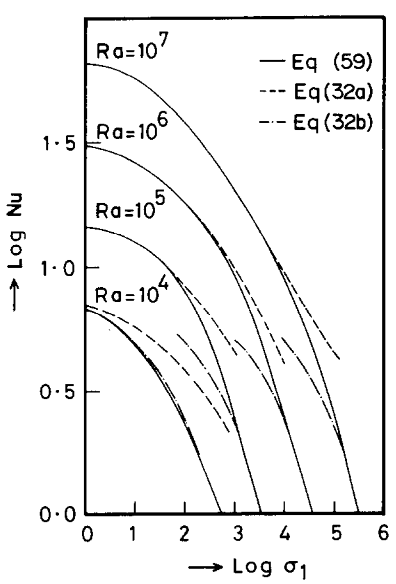

FIG. 3. Log Nu vs log σ_1 for four values of Ra, with $a = \pi$.

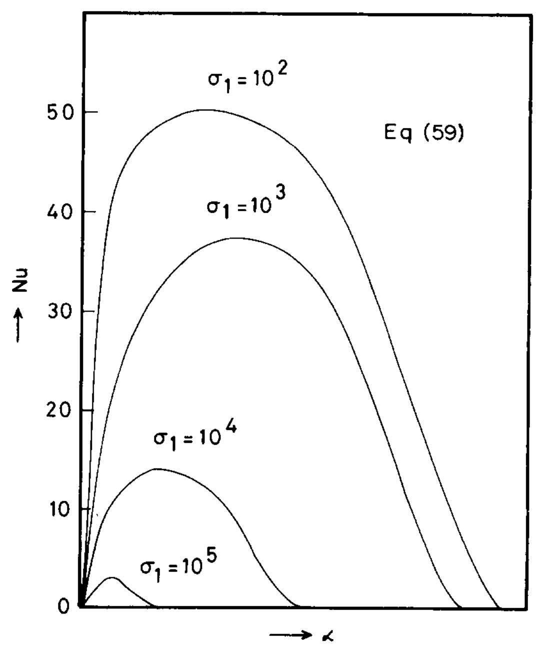

FIG. 5. Na vs α for $Ra = 10^7$.

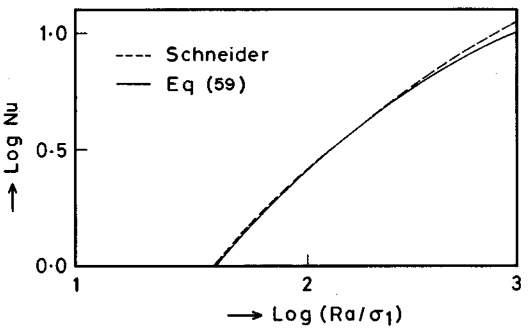

FIG. 6. Log Nu vs log Ra/σ_1.

presence of a magnetic field [15] without a porous medium. The results are found to be in agreement with those obtained by numerical methods [6] thus demonstrating the fact that the above method can successfully be employed to treat the problem of thermal convection over a wide range of Ra and σ_1.

The fundamental result obtained in all cases is that the Nusselt number varies as $Ra^{1/3}$ for a given σ_1. Earlier such studies carried out by Masuoka using Darcy's equation [8] led us to believe that Nu is proportional to Ra/σ_1 for porous medium when $Ra_1 \geqslant 2Ra_c$.

Figure 4 shows the variation of Nu with Ru for different values of σ_1. We note that as σ_1 increases the magnitude of W decreases and hence the convection is suppressed.

Figure 5 shows the variation of Nu with α for $Ra = 10^7$. The value of α which maximizes Nu is different for different values of σ_1.

Figure 6 shows the variation of Nu vs Ra/σ_1 calculated from equation (59). The results are in agreement with the experimental results of Schneider for the whole range of Ra/σ_1 and with those of Masuoka [8] for a lower range of Ra/σ_1.

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SOLUTIONS ASYMPTOTIQUES POUR LA CONVECTION THERMIQUE NON LINEAIRE DANS LES MILIEUX POREUX

Résumé—La convection thermique non linéaire dans un milieu poreux saturé est étudiée sous différentes approximations. Des solutions asymptotiques pour des frontières libres sont obtenues dans deux cas limites: Ra grand, $\sigma_1 = O(1)$ et $O(Ra)$ en utilisant la méthode des développements asymptotiques. La méthode analytique d'approximation est aussi utilisée pour étudier le problème de convection dans un large domaine de Ra et de σ_1. Les résultats de l'analyse montre que le flux thermique varie comme $Ra^{1/3}$ pour σ_1 donné. Les résultats sont en accord avec les résultats expérimentaux de Schneider.

ASYMPTOTISCHE LÖSUNGEN FÜR NICHTLINEARE THERMISCHE KONVEKTION IN PORÖSEN MEDIEN

Zusammenfassung—Nichtlineare thermische Konvektion in einem gesättigten porösen Medium wurde unter verschiedenen Annahmen untersucht. Asymptotische Lösungen für freie Grenzen wurden für zwei Grenzfälle, Ra groß, $\sigma_1 = O(1)$ und $O(Ra)$, erhalten, indem man die Methode der angepaßten asymptotischen Reihenentwicklung benutzt. Diese analytische Näherungsmethode wird ebenso benutzt, um das Konvektionsproblem über einen großen Bereich von Ra und σ_1 zu untersuchen. Die Ergebnisse dieser Untersuchung zeigen, daß sich für gegebenes σ_1 der Wärmeübergangskoeffizient mit $Ra^{1/3}$ ändert. Die Ergebnisse stimmen mit Versuchsergebnissen von Schneider überein.

АСИМПТОТИЧЕСКИЕ РЕШЕНИЯ ДЛЯ НЕЛИНЕЙНОЙ ТЕПЛОВОЙ КОНВЕКЦИИ В ПОРИСТЫХ СРЕДАХ

Аннотация—С помощью различных приближений исследуется нелинейная тепловая конвекция в пропитанной жидкостью пористой среде. Методом сращиваемых асимптотических разложений получены решения для свободных границ в двух предельных случаях: при большом значении числа Ra для $\sigma_1 = O(1)$ и $O(Ra)$. Для исследования процесса конвекции в широком диапазоне изменения чисел Ra и σ_1 использован также метод аналитических аппроксимаций. Результаты анализа показывают, что плотность потока тепла изменяется пропорционально $Ra^{1/3}$ при указанном выше значении σ_1. Результаты согласуются с экспериментальными данными Шнайдера.